Weak solutions to the stochastic porous media equation via Kolmogorov equations: the degenerate case

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Abstract. A stochastic version of the porous medium equation with coloured noise is studied. The corresponding Kolmogorov equation is solved in the space $L^2(H,\nu)$ where ν is an infinitesimally excessive measure. Then a weak solution is constructed.

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1 Introduction

The porous medium equation

$$\frac{\partial X}{\partial t} = \Delta(\Psi(X)), \quad m \in \mathbb{N}, \tag{1.1}$$

on a bounded open set $D \subset \mathbb{R}^d$ with Dirichlet boundary conditions for the Laplacian Δ and with Ψ in a large class of functions has been studied extensively (see e.g. [1], [2, Section 4.3]). Recently, there has been also several papers on the stochastic version of (1.1), i.e.

$$dX(t) = \Delta(\Psi(X(t))dt + \sqrt{C}\, dW(t), \quad t \geq 0, \tag{1.2}$$

(cf. [10], [11], [6] and [3]).

In this paper we continue the study of the stochastic partial differential equation (SPDE) (1.2). Before we describe our new results precisely, let us fix some notation and our exact conditions.

The appropriate state space on which we consider (1.2) is $H := H^{-1}(D)$, i.e., the dual of the Sobolev space $H_0^1 := H_0^1(D)$, with inner product $\langle \cdot, \cdot \rangle_H$. Below we shall use the standard $L^2(D)$-dualization $\langle \cdot, \cdot \rangle_H$ between $H_0^1(D)$ and $H = H^{-1}(D)$ induced by the embeddings

$$H_0^1(D) \subset L^2(D)' = L^2(D) \subset H^{-1}(D) = H$$

without further notice. Then for $x \in H$ one has

$$|x|_H^2 = \int_D ((-\Delta)^{-1}x)(\xi)\, x(\xi) d\xi.$$

Let $(W_t)_{t\geq 0}$ be a cylindrical Brownian motion in H and let C be a positive definite bounded operator on H of trace class. To be more concrete below we assume:

$(H1)$ *There exist numbers $\lambda_k \in [0,\infty)$, $k \in \mathbb{N}$ such that for the eigenbasis $\{e_k |\ k \in \mathbb{N}\}$ of Δ in H (with Dirichlet boundary conditions) we have $Ce_k = \lambda_k e_k$ for all $k \in \mathbb{N}$.*

$(H2)$ *For $\alpha_k := \sup_{\xi\in D} |e_k(\xi)|^2$, $k \in \mathbb{N}$, we have $K := \sum_{k=1}^{\infty} \alpha_k\lambda_k < +\infty$.*

$(H3)$ *There exist $\Psi \in C^1(\mathbb{R}), r \in (1,\infty), \kappa_0, \kappa_1, C_1 > 0$ such that*

$$\kappa_0|s|^{r-1} \leq \Psi'(s) \leq \kappa_1|s|^{r-1} + C_1 \quad \text{for all } s \in \mathbb{R} \text{ (cf. [6])}.$$

Our general aim in studying SPDE (1.2) is to construct a strong Markov weak solution for (1.2), i.e., a solution in the sense of the corresponding martingale problem (see [21] for the finite dimensional case), at least for a large set $\overline{H}$ of starting points in H which is invariant for the process, i.e., with probability one $X_t \in \overline{H}$ for all $t \geq 0$. We follow the strategy first presented in [17] (and already carried out in cases with bounded C^{-1} in [9]). That is, first we want to construct a solution to the corresponding Kolmogorov equations in $L^2(H,\mu)$ for suitably chosen reference measures μ (see below), and then a strong Markov process with continuous sample paths having transition probabilities given by that solution to the Kolmogorov equations. As in [9] we also aim to prove that this process is for μ-a.e. starting point $x \in \overline{H}$ a unique (in distribution) continuous Markov process whose transition semigroup consists of continuous operators on $L^2(H,\mu)$, which is e.g. the case if μ is one of its excessive measures.

Before we summarize the specific new results of this paper in relation to those obtained in [10],[11], [6], let us describe this programme more precisely.

Applying Itô's formula (at a heuristic level) to (1.2) one finds what the corresponding Kolmogorov operator, let us call it N_0, should be, namely

$$N_0\varphi(x) = \frac{1}{2}\sum_{k=1}^{\infty} \lambda_k D^2\varphi(e_k, e_k) + D\varphi(x)(\Delta(\Psi(x))), \quad x \in H, \tag{1.3}$$

where $D\varphi$, $D^2\varphi$ denote the first and second Fréchet derivatives of $\varphi : H \to \mathbb{R}$. So, we take $\varphi \in C_b^2(H)$.

In order to make sense of (1.3) one needs that $\Delta(\Psi(x)) \in H$ at least for "relevant" $x \in H$. Here one clearly sees the difficulties since $\Psi(x)$ is, of course, not defined for any Schwartz distribution in $H = H^{-1}$, not to mention that it will not be in $H_0^1(D)$. So, a way out of this is to think about "relevant" $x \in H$. Our approach to this is first to look for an invariant measure for the solution to equation (1.2) which can now be defined "infinitesimally"(cf. [5]) without having a solution to (1.2) as a solution to the equation

$$N_0^*\mu = 0 \tag{1.4}$$

with the property that μ is supported by those $x \in H$ for which $\Psi(x)$ makes sense and $\Delta(\Psi(x)) \in H$. Equation (1.4) is a short form for

$$N_0\varphi \in L^1(H,\mu) \quad \text{and} \quad \int_H N_0\varphi d\mu = 0 \quad \text{for all } \varphi \in C_b^2(H). \tag{1.5}$$

Any invariant measure for any solution of (1.2) in the classical sense will satisfy (1.4). Then we can analyze N_0, with domain $C_b^2(H)$ in $L^2(H,\mu)$, i.e., solve the Kolmogorov equation

$$\frac{dv}{dt} = \overline{N_0}v, \quad v(0,\cdot) = f, \tag{1.6}$$

for the closure $\overline{N_0}$ of N_0 on $L^2(H,\mu)$ and initial condition $f \in L^2(H,\mu)$. This means, we have to prove that $\overline{N_0}$ generates a C_0-semigroup $T_t = e^{t\overline{N_0}}$ on $L^2(H,\mu)$, i.e., that $(N_0, C_b^2(H))$ is essentially m-dissipative on $L^2(H,\mu)$.

Subsequently, we have to show that $(T_t)_{t\geq 0}$ is given by a semigroup of probability kernels $(p_t)_{t\geq 0}$ (i.e., $p_t f$ is a μ-version of $T_t f \in L^2(H,\mu)$ for any $t \geq 0$ and any bounded measurable function $f\colon\ H \to \mathbb{R}$) and such that there exists a strong Markov process with continuous sample paths in H whose transition function is $(p_t)_{t\geq 0}$. Then, by definition, this Markov process will solve the martingale problem corresponding to (1.2).

The existence of solutions to (1.4) (even for more general SPDE than (1.2)) was proved in [6] (the method was based essentially on finite dimensional approximations), generalizing earlier results from [10]. We shall restate the precise theorem in §2 below.

In [10] in the special case when

$$\Psi(s) := \alpha s + s^m, \quad s \in \mathbb{R}, \tag{1.7}$$

for $m \in \mathbb{N}$, m odd, and $\alpha > 0$, the remaining part of the above programme was carried out. The specially interesting "degenerate" case $\alpha = 0$ in (1.7) was, however, not covered.

In this paper we shall improve these results in an essential way. First, we shall construct a solution to the Kolmogorov equation (1.6) for Ψ as in (H3), hence including the case $\alpha = 0$ in (1.7). More precisely, we identify a whole class $\mathcal{M}$ of reference measures, called infinitesimally excessive measures, which includes all measures solving (1.4) so that for all $\nu \in \mathcal{M}$ we can construct a solution to the Kolmogorov equation (1.6) in $L^2(H,\nu)$ for Ψ as in (H3), hence including the degenerate case $\alpha = 0$, in (1.7). The main tool employed here is the Yosida approximation for the nonlinear maximal dissipative mapping $\Delta(\Psi)$, as a map in H^{-1} with domain H_0^1. In particular, we thus clarify that in case the nonlinearity of SPDE (1.2) is maximal dissipative, the issue of proving the existence of infinitesimally invariant measures μ for N_0 and the issue of essentially maximal dissipativity of the operator $(N_0, C_b^2(H))$ on $L^2(H,\nu)$ can be separated completely. That is, the latter does not depend in particular on how one constructs a solution to (1.4) and which solution is chosen as a reference measure.

Second, we shall construct the said Markov process which weakly solves SPDE (1.2) for general Ψ as in (H3); i.e., without any nondegeneracy assumptions. Furthermore, we prove that for $d = 1$ and specifically chosen C (cf. condition (H.4) in §5) the Markov process is strong Feller.

The organization of this paper is as follows. In §2 we summarize all relevant results about infinitesimal invariant measures μ for N_0 from [6] and [10]. Then we define the mentioned class $\mathcal{M}$ of references measures ν and prove that for some $\lambda > 0$, $(\lambda - N_0, C_b^2(H))$ is dissipative on $L^2(H,\nu)$, hence $(N_0, C_b^2(H))$ is closable on $L^2(H,\nu)$.

§3 is devoted to the Yosida approximations. In §4 we prove that for all $\nu \in \mathcal{M}$ the closure of $(N_0, C_b^2(H))$ on $L^2(H,\nu)$ generates a C_0-semigroup on $L^2(H,\nu)$ solving (1.6). §5 is devoted to the existence and uniqueness of a Markov process solving SPDE (1.2) in the sense of a martingale problem, and, in case $d = 1$, to its strong Feller property on $supp\,\nu$. In §6 under weak additional conditions we prove that if ν is the solution of (1.4) constructed in [6], then $supp\,\nu = H$, i.e. ν charges any non-empty open set of H.

Finally, we would like to mention that we think that it should be also possible to prove the existence and uniqueness of a strong solution for (1.2). A corresponding paper of the last named author jointly with B. Rozovskii is in preparation.

2 Infinitesimal invariance and a large class of references measures

We first note that $N_0\varphi(x)$ is well defined for $\varphi \in C_b^2(H)$ if x belongs to the set

$$H_\Psi := \{x \in L^2(D) : \ \Psi(x) \in H_0^1\}. \tag{2.1}$$

We also define for $r > 1$

$$H_{0,r}^1 := \{x \in L^2(D) : \ |x|^r \text{sign } x \in H_0^1\}.$$

Now we recall the following result from [6] (see Theorem 1.1 and Corollary 1.1 ibid).

Theorem 2.1. *Assume that (H1)–(H3) hold. Then there exists a probability measure μ on H which is infinitesimally invariant for N_0 in the sense of* (1.5). *Furthermore,*

$$\int_H \int_D |\nabla(\Psi(x))(\xi)|^2 d\xi \mu(dx) < +\infty \tag{2.2}$$

and

$$\int_H \int_D |\nabla(|x|^{\frac{r+1}{2}} \text{ sign } x)(\xi)|^2 d\xi \mu(dx) < +\infty. \tag{2.3}$$

In particular, $\mu(H_\Psi \cap H^1_{0,\frac{r+1}{2}}) = 1$.

Remark 2.2. It was also shown in [6, Lemma 1] that $H_\Psi \subset H_{0,r}^1$. So, (2.2) implies that

$$\int_H \int_D |\nabla(|x|^r \text{ sign } x)(\xi)|^2 d\xi \mu(dx) < +\infty.$$

Therefore, by Poincaré's inequality, $H_\Psi \subset L^{2r}(D)$ and

$$\int_H \int_D |x|^{2r}(\xi) d\xi \mu(dx) < +\infty.$$

By Theorem 2.1, $N_0\varphi$ is μ-a.e. defined for all $\varphi \in C_b^2(H)$. All subsequent results in this paper are valid for the larger class of measures $\mathcal{M}$ on H which contains all infinitesimally invariant measures for N_0 and consists of all probability measures ν on H which satisfy (2.2) and for which there exists $\lambda_\nu \in (0, \infty)$ such that

$$\int_H N_0\varphi d\nu \leq \lambda_\nu \int_H \varphi d\nu \quad \text{for all } \varphi \in C_b^2(H) \text{ with } \varphi \geq 0 \ \ \nu\text{-a.e.} \tag{2.4}$$

The elements in $\mathcal{M}$ are called *infinitesimally excessive measures.*

Lemma 2.3. *Let $\nu \in \mathcal{M}$ and $\varphi \in C_b^2(H)$ be such that $\varphi = 0$ ν-a.e. Then $N_0\varphi = 0$ ν-a.e.*

Proof. The proof is analogous to that of Lemma 3.1 in [6] (see also [18, Proposition 4.1]). □

We would like to emphasize that so far we have not been able to show that $\mu(U) > 0$ (for μ as in Theorem 2.1) for any open non empty set $U \subset H$. So, Lemma 2.3 is crucial to consider N_0 as an operator on $L^2(H, \mu)$ with domain equal to the μ-classes determined by $C_b^2(H)$, again denoted by $C_b^2(H)$. The same holds for any $\nu \in \mathcal{M}$.

Proposition 2.4. *Assume that $(H1)$–$(H3)$ hold and let $\nu \in \mathcal{M}$. Then:*

(i) *$N_0\varphi \in L^2(H, \nu)$ for $\varphi \in C_b^2(H)$.*

(ii) *$(\frac{1}{2}\lambda_\nu - N_0, C_b^2(H))$ is dissipative on $L^2(H, \nu)$, i.e.,*

$$\|\lambda(\lambda + \frac{1}{2}\lambda_\nu - N_0\varphi\|_{L^2(H,\nu)} \geq \|\varphi\|_{L^2(H,\nu)} \text{ for all } \varphi \in C_b^2(H).$$

In particular, $(N_0, C_b^2(H))$ is closable on $L^2(H, \nu)$, its closure being denoted by $(N_2, D(N_2))$.

Proof. (i) We note that

$$\int_D |\nabla\Psi(x)|^2(\xi) d(\xi) = |\Delta\Psi(x)|_H^2.$$

Hence the assertion follows by (2.2).

(ii) This follows from [14, Appendix B, Lemma 1.8]. □

3 Yosida approximations

For completeness we recall the definition and basic properties of the Yosida approximation of an m-dissipative map $F\colon D(F) \subset H \to H$. The latter means that

$$\langle F(x) - F(y), x - y\rangle_H \leq 0 \quad \text{for all } x, y \in D(F) \tag{3.1}$$

and

$$(\lambda I - F)(D(F)) = H \quad \text{for all } \lambda > 0, \tag{3.2}$$

where I denotes the identity operator on H. For $\varepsilon > 0$ let

$$J_\varepsilon := (I - \varepsilon F)^{-1}. \tag{3.3}$$

Note that by (3.1) $I - \varepsilon F : D(F) \to H$ is one–to–one. Then J_ε is Lipschitz continuous with constant 1, hence so is

$$F_\varepsilon := \frac{1}{\varepsilon}(J_\varepsilon - I) \tag{3.4}$$

with constant $\frac{1}{\varepsilon}$. F_ε is called *Yosida approximation* of F. It has the following properties (cf. e.g. [2], [8] or [19]):

$$\lim_{\varepsilon\to 0} F_\varepsilon(x) = F(x), \quad x \in D(F), \tag{3.5}$$

$$|F_\varepsilon(x)|_H \le |F(x)|_H,, \quad x \in D(F),\ \varepsilon > 0, \tag{3.6}$$

$$|F_\varepsilon(x)|_H \uparrow 1_{D(F)}(x)|F(x)| + \infty \cdot 1_{H\setminus D(F)}(x), \quad \text{as } \varepsilon \downarrow 0,\ x \in H, \tag{3.7}$$

$$\langle F_\varepsilon(x), F(x)\rangle_H \le -|F_\varepsilon(x)|_H^2, \quad x \in D(F). \tag{3.8}$$

The following is well known, see e.g. [2, Chapter 2, Proposition 2.12] and for the original proof [7].

Proposition 3.1. *Assume* $(H3)$ *holds. Then* $F := \Delta\Psi$ *with domain* $D(F)\colon = H_\Psi$ *is m-dissipative on* H.

4 Essential maximal dissipativity of N_0 on $L^2(H,\nu)$

Below, F_ε, $\varepsilon > 0$, shall always denote the Yosida approximation to $(\Delta\Psi, H_\Psi)$. We need a further regularization and, therefore, define for $\beta > 0$

$$F_{\varepsilon,\beta}(x) := \int_H e^{\beta B} F_\varepsilon(e^{\beta B}x + y) N_{\frac{1}{2}\,B^{-1}(e^{2\beta B}x - I)}, \quad x \in H, \tag{4.1}$$

where $B\colon D(B) \subset H \to H$ is a self-adjoint negative definite operator such that B^{-1} is of trace class. Then obviously $F_{\varepsilon,\beta}$ is dissipative of class C^∞, and has bounded derivatives of all orders. Furthermore,

$$\lim_{\beta\to 0} F_{\varepsilon,\beta}(x) = F_\varepsilon(x), \quad x \in H, \tag{4.2}$$

(see [12, Theorem 9.19]) and, since F_ε is Lipschitz, there exists $c_\varepsilon \in (0,\infty)$ such that

$$|F_{\varepsilon,\beta}(x)| \le c_\varepsilon(1 + |x|_H), \quad x \in H, \quad x \in H, \beta > 0. \tag{4.3}$$

Now consider the approximating stochastic equation

$$dX(t) = F_{\varepsilon,\beta}(X(t))dt + \sqrt{C}\, dW(t) \tag{4.4}$$

It is well known (see [12]) that for any initial condition $x \in H$ equation (4.4) has a unique solution $X_{\varepsilon,\beta}(\cdot, x)$ and that for $\lambda > 0$ and $f \in C_b^2(H)$

$$\varphi_{\varepsilon,\beta}(x) = \int_0^\infty e^{-\lambda t}\mathbb{E}[f(X_{\varepsilon,\beta}(t,x))]dt, \quad x \in H, \tag{4.5}$$

is in $C_b^2(H)$ and solves the equation

$$f(x) = \lambda\varphi_{\varepsilon,\beta}(x) - \frac{1}{2}\sum_{k=1}^\infty \lambda_k D^2\varphi_{\varepsilon,\beta}(x)(e_k, e_k) + D\varphi_{\varepsilon,\beta}(x)(F_{\varepsilon,\beta}(x)), \tag{4.6}$$

(see [13, Chapter 5.4]). We have moreover for all $h \in H$,

$$D\varphi_{\varepsilon,\beta}(x)(h) = \int_0^{+\infty} e^{-\lambda t}\mathbb{E}[Df(X_{\varepsilon,\beta}(t,x))(D_x X_{\varepsilon,\beta}(t,x)h)]dt. \tag{4.7}$$

For any $h \in H$ we set $\eta^h_{\varepsilon,\beta} := D_x X_{\varepsilon,\beta}(t,x)$. Then we have

$$\begin{cases} \frac{d}{dt}\,\eta^h_{\varepsilon,\beta}(t,x) = DF_{\varepsilon,\beta}(X_{\varepsilon,\beta}(t,x))\eta^h_{\varepsilon,\beta}(t,x) \\ \\ \eta^h_{\varepsilon,\beta}(0,x) = h. \end{cases} \tag{4.8}$$

Multiplying both sides of equation (4.8) by $\eta^h_{\varepsilon,\beta}(t,x)$, integrating with respect to t and taking the dissipativity of $DF_{\varepsilon,\beta}$ into account, we find

$$|\eta^h_{\varepsilon,\beta}(t,x)|^2 \le |h|^2. \tag{4.9}$$

Consequently by (4.7) it follows that

$$|D\varphi_{\varepsilon,\beta}(x)|_{H_0^1} \le \frac{1}{\lambda}\,\|Df\|_0,\ x \in H, \tag{4.10}$$

where $\|\cdot\|_0$ denotes the sup norm.

Now we can prove the following result.

Theorem 4.1. *Assume that (H1)–(H3) hold and let $\nu \in \mathcal{M}$. Then $(N_0, C_b^2(H))$ is essentially m-dissipative on $L^2(H,\nu)$, i.e., its closure $(N_2, D(N_2))$ is maximal dissipative on $L^2(H,\nu)$.*

Proof. Let $f \in C_b^2(H)$ and let $\varphi_{\varepsilon,\beta}$ be the solution to equation (4.6). Then $\varphi_{\varepsilon,\beta} \in C_b^2(H)$ and we have

$$\lambda\varphi_{\varepsilon,\beta} - N_0\varphi_{\varepsilon,\beta} = f + D\varphi_{\varepsilon,\beta}(F_{\varepsilon,\beta} - \Delta\Psi). \tag{4.11}$$

We claim that

$$\lim_{\varepsilon\to 0}\lim_{\beta\to 0} D\varphi_{\varepsilon,\beta}(F_{\varepsilon,\beta} - \Delta\Psi) = 0 \ \text{ in } L^2(H,\nu).$$

In fact by (4.10) it follows that

$$I_{\varepsilon,\beta} := \int_H |D\varphi_{\varepsilon,\beta}(F_{\varepsilon,\beta} - \Delta\Psi)|^2_{H_0^1} d\nu \le \frac{1}{\lambda^2}\,\|Df\|_0^2 \int_H |F_{\varepsilon,\beta} - \Delta\Psi|^2_H d\nu. \tag{4.12}$$

Letting $\beta \to 0$ we conclude by (4.3) that

$$\limsup_{\beta \to 0} I_{\varepsilon,\beta} \leq \frac{1}{\lambda^2} \|Df\|_0^2 \int_H |F_\varepsilon - \Delta\Psi|_H^2 d\nu.$$

Since ν verifies (2.2) by assumption, the claim now follows, in view of the dominated convergence theorem, from (3.6)–(3.7) with $F := \Delta\Psi$.

Hence we have proved that

$$\lim_{\varepsilon \to 0} \lim_{\beta \to 0} (\lambda - N_0)\varphi_{\varepsilon,\beta} = f \text{ in } L^2(H,\nu).$$

Therefore the closure of the range of $\lambda - N_0$ includes $C_b^2(H)$ which is dense in $L^2(H,\nu)$. By the Lumer–Phillips theorem it follows that N_2 is maximal–dissipative as required. □

As a consequence of the proof of Theorem 4.1 we have:

Corollary 4.2. *Assume that (H1)–(H3) hold and let $\nu \in \mathcal{M}$. Define a C_0-semigroup*

$$P_t := e^{tN_2}, \quad t \geq 0,$$

on $L^2(H,\nu)$ (which exists by Theorem 4.1). Then

(i) *$v(t,\cdot) := P_t f$, $t > 0$, solves (1.6) for the initial datum $f \in D(N_2)$.*

(ii) *$(P_t)_{t\geq 0}$ is Markovian, i.e., $P_t 1 = 1$ and $P_t f \geq 0$ for all nonnegative $f \in L^2(H,\nu)$ and all $t \geq 0$.*

(iii) *Let $f \in L^2(H,\nu)$, f nonnegative, and $t > 0$. Then*

$$\int_H P_t f d\nu \leq e^{\lambda_\nu t} \int_H f d\nu. \tag{4.13}$$

Proof. (i) The assertion follows by the definition of P_t, $t \geq 0$.

(ii) By [14, Appendix B, Lemma 1.9] P_t is positivity preserving. Since $1 \in C_b^2(H)$ and $N_0 1 = 0$, it follows that $P_t 1 = 1$ for all $t \geq 0$.

(iii) We first note that since $C_b^2(H)$ is dense in $D(N_2)$ with respect to the graph norm given by N_2, it follows by Theorem 4.1 and (2.4) that

$$\int_H N_2 f d\nu \leq \lambda_\nu \int_H f d\nu, \quad \text{for all } f \in D(N_2) \text{ with } f \geq 0 \ \nu\text{-a.e.} \tag{4.14}$$

So, if $f \in C_b^2(H)$ ($\subset D(N_2)$), $f \geq 0$, then $P_t f \in D(N_2)$ and $P_t f \geq 0$ ν-a.e. Hence (4.14) and assertion (i) imply that

$$\frac{d}{dt} \int_H P_t f d\nu = \int_H N_2 P_t f d\nu \leq \lambda_\nu \int_H P_t f d\nu.$$

So, by Gronwall's lemma (4.13) follows for $f \in C_b^2(H)$, $f \geq 0$. But since any nonnegative $f \in L^2(H,\nu)$ can be approximated by nonnegative functions in $C_b^2(H)$ in $L^2(H,\nu)$, assertion (iii) follows. □

5 Existence of a weak solution of SPDE (1.2)

This section generalizes all results of §4 in [11] in an essential way. However, parts of it are very similar. We, nevertheless, include a complete presentation below for the reader's convenience.

Theorem 5.1 (Existence). *Assume that $(H1)$–$(H3)$ hold and, in addition, that $r \geq 2$. Let $\nu \in \mathcal{M}$. Then*

(i) *There exists a conservative strong Markov process*

$$\mathbb{M} = \Big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in H}\Big)$$

on H with continuous sample paths such that for its transition semigroup $(p_t)_{t\geq 0}$ defined by

$$p_t f(x) := \int_H f(X_t) d\mathbb{P}_x, \quad t \geq 0, x \in H,$$

where $f\colon H \to \mathbb{R}$ is bounded $\mathcal{B}(H)$-measurable, we have that $p_t f$ is a ν-version of $e^{tN_2} f$, $t > 0$. Furthermore, if $f \geq 0$, one has

$$\int_H p_t f d\nu \leq e^{\lambda_\nu t} \int_H f d\nu \quad \text{for all } t \geq 0,$$

i.e., ν is an excessive measure for $\mathbb{M}$.

(ii) *There exists $\overline{H} \in \mathcal{B}(H)$ such that $\nu(\overline{H}) = 1$, for all $x \in \overline{H}$ one has*

$$\mathbb{P}_x[X_t \in \overline{H} \quad \forall\, t \geq 0] = 1,$$

and for any probability measures ρ on $(H, \mathcal{B}(H))$ with $\rho(\overline{H}) = 1$, the process

$$\varphi(X_t) - \int_0^t N_0\varphi(X_s) ds, \quad t \geq 0,$$

is an $(\mathcal{F}_t)$-martingale under $\mathbb{P}_\rho := \int_{\overline{H}} \mathbb{P}_x \rho(dx)$ for all $\varphi \in C_b^2(H)$ and one has $\mathbb{P}_\rho \circ X_0^{-1} = \rho$.

Theorem 5.2 (Uniqueness). *Assume that $(H1)$–$(H3)$ hold and, in addition, that $r \geq 2$. Let $\nu \in \mathcal{M}$. Suppose that*

$$\mathbb{M}' = \Big(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, (X'_t)_{t\geq 0}, (\mathbb{P}'_x)_{x\in H}\Big)$$

is a continuous Markov process on H whose transition semigroup $(p'_t)_{t\geq 0}$ consists of continuous operators on $L^2(H, \mu)$ with locally (in t) uniformly bounded operator norm (which is e.g. the case if ν is also an excessive measure for $\mathbb{M}'$). If $\mathbb{M}'$ satisfies assertion (ii) *of Theorem* 5.1 *for $\rho := \nu$, then for ν-a.e. $x \in H$, one has $p'_t(x, dy) = p_t(x, dy)$ for all $t \geq 0$ (where p_t is as in Theorem* 5.1-(i)*), i.e., $\mathbb{M}'$ has the same finite dimensional distributions as $\mathbb{M}$ for ν-a.e. starting point.*

We shall only prove Theorem 5.1-(i). The remaining parts are proved in exactly the same way as Theorem 7.4-(ii), Proposition 8.2 and Theorem 8.3 in [9] with the only exception that because we do not know whether $(p_t)_{t\geq 0}$ is Feller, all statements can only be proved ν-a.e. So we do not want to repeat them here.

Our proof of Theorem 5.1-(i) is based on the theory of generalized Dirichlet forms developed in [20]. Indeed, by Corollary 4.2, $(N_2, D(N_2))$ is a Dirichlet operator in the sense of [16], [20]. Hence by [20, Proposition I.4.6]

$$\mathcal{E}(u,v) := \begin{cases} (u,v)_{L^2(H,\nu)} - (N_2 u, v)_{L^2(H,\nu)}, & u \in D(N_2),\ v \in L^2(H,\nu), \\ \\ (u,v)_{L^2(H,\nu)} - (N_2^* v, u)_{L^2(H,\nu)}, & u \in L^2(H,\nu),\ v \in D(N_2^*), \end{cases}$$

is a generalized Dirichlet form on $L^2(H,\nu)$ in the sense of [20, Definition I.4.8] with

$$\mathcal{F} := \big(D(N_2), \|N_2 \cdot\|_{L^2(H,\nu)} + \|\cdot\|_{L^2(H,\nu)}\big)$$

and with coercive part $\mathcal{A}$ identically equal to 0.

We emphasize here that the theory of generalized Dirichlet forms, in contrast to earlier versions (cf. e.g. [15], [16]), does not require any symmetry or sectoriality of the underlying operators. We refer to [20] for an excellent exposition. As is well known to the experts on potential theory on L^2-spaces (and as is clearly presented in [20]), the following two main ingredients are needed.

(a) There exists a core C of $(N_2, D(N_2))$ which is an algebra consisting of functions having (quasi) continuous ν-versions.

(b) The capacity determined by $(N_2, D(N_2))$ is tight.

(a) follows from the essential m-dissipativity of N_0 on $C_b^2(H)$ proved in the previous section, so we can take $C := C_b^2(H)$. This is exactly why essential m-dissipativity is so important for probability theory, in particular, Markov processes. Before we prove (b) we recall the necessary definitions.

Let

$$G_\lambda^{(2)} := (\lambda - N_2)^{-1}, \quad \lambda > 0,$$

be the resolvent corresponding to N_2. A function $u \in L^2(H,\nu)$ is called 1-*excessive* if $u \geq 0$ and $\lambda G_{1+\lambda} u \leq u$ for all $\lambda > 0$. For an open set $U \subset H$ define

$$e_U := \inf\{u \in L^2(H,\nu) |\ u \text{is 1-excessive},\ u \geq 1_U\ \nu\text{-a.e.}\},$$

(cf. [20, Proposition III.1.7 (ii)]), and the 1-*capacity* of U by

$$\text{Cap}\, U := \int_H e_U d\nu.$$

(cf. [20, Definition III.2.5 with $\varphi \equiv 1$]). Cap is called *tight* if there exist increasing compact sets K_n, $n \in \mathbb{N}$, such that for $K_n^c := H \setminus K_n$ one has

$$\lim_{n\to\infty} \text{Cap}\,(K_n^c) = 0.$$

Once we have proved this, i.e., have proved (b), Theorem 5.1-(i) follows from [20, Theorem IV.2.2]. Indeed, in our situation, according to (a) and [20, Proposition IV.2.1], the requirement in [20, Theorem IV.2.2] that quasi-regularity holds is equivalent to (b) and condition $D3$ in [20, Theorem IV.2.2].

Remark 5.3. We mention here that in Theorem 5.1 we do not state all facts known about $\mathbb{M}$; e.g., it is also proved in [20, Theorem IV.2.2, see also Definition IV 1.4] that all "ν-a.e." statements can be replaced by "quasi everywhere" (with respect to Cap) statements and that

$$x \mapsto \int_0^{+\infty} e^{-\lambda t} p_t f(x) dt$$

is Cap-quasi-continuous. Furthermore, [20, Theorem IV.2.2] only claims that $\mathbb{M}$ has cadlag paths, but a similar proof as that in [16, Chapter V, Sect. 1] gives indeed continuous paths because N_2 is a local operator.

To prove (b) it is enough to find a 1-excessive function $u\colon H \to \mathbb{R}^+$ so that for each $n \in \mathbb{N}$ the level set $\{u \leq n\}$ is contained in the union of a compact set $K_n \subset H$ and a ν-zero set, because then $e_{K_n^c} \leq \frac{1}{n}\, u$ ν-a.e., hence

$$\text{Cap}\,(K_n^c) \leq \frac{1}{n} \int_H u d\nu \to 0, \quad \text{as } n \to \infty. \tag{5.1}$$

So, the proof of Theorem 5.1-(i) is completed by Proposition 5.4 below, since closed balls in $L^2(D)$ are compact in H. Before we can formulate it, we need to introduce the resolvent generated by N_0 on $L^1(H,\nu)$. To this end we note that by (2.4) $(N_0, C_b^2(H))$ is also dissipative on $L^1(H,\nu)$ (cf. e.g. [14, Appendix B, Lemma 1.8]), hence closable. We recall that $(\lambda - N_0)(C_b^2(H))$ is dense in $L^2(H,\nu)$ (by the proof of Theorem 4.1 above), hence also dense in $L^1(H,\nu)$, so analogously $(N_1, D(N_1))$ generates a C_0 semigroup $(e^{tN_1})_{t\geq 0}$ of contractions on $L^1(H,\nu)$ and we can consider the corresponding resolvent

$$G_\lambda^{(1)} := (\lambda - N_1)^{-1}, \quad \lambda > 0,$$

Clearly, $G_\lambda^{(1)} = G_\lambda^{(2)}$ on $(\lambda - N_0)(C_b^2(H))$, hence

$$G_\lambda^{(1)} f = G_\lambda^{(2)} f \quad \text{for all } \lambda > 0,\ f \in L^2(H,\nu). \tag{5.2}$$

Define

$$\overline{\Psi}(t) := \int_0^t \Psi(s) ds, \quad t \in \mathbb{R},$$

and

$$\Phi(x) := \begin{cases} \int_D \overline{\Psi}(x(\xi)) d\xi, & x \in H_\Psi \\ \\ +\infty & \text{otherwise.} \end{cases}$$

By (H3) $\overline{\Psi}$ is convex and since $r > 1$, (H3) also implies that for all $s \in \mathbb{R}$

$$\begin{aligned} 0 \leq \frac{\kappa_0}{r(r+1)}\, |s|^{r+1} \leq \overline{\Psi}(s) &\leq \frac{C_1}{2}\, |s|^2 + \frac{\kappa_1}{r(r+1)}\, |s|^{r+1} \\ &\leq \left[\frac{C_1}{2} + \left(\frac{C_1}{2} + \frac{\kappa_1}{\kappa_0(r+1)}\right) |\Psi(s)|\right] |s|. \end{aligned} \tag{5.3}$$

Hence, it follows by Remark 2.2 that $\Phi \in L^1(H,\nu)$. Recall that by (2.2) $|\Delta\Psi|_H^2 \in L^1(H,\nu)$.

Proposition 5.4. *Consider the situation of Theorem* 5.1. *Then*

(i) *There exists* $c > 0$ *such that*

$$c|x|^{r+1}_{L^2(D)} \leq G_1^{(1)}(\Phi + |\Delta\Psi|_H^2)(x) =: g(x) \ (\geq 0) \ \textit{ for } \nu\textit{-a.e. } x \in H.$$

(ii) *The function* $g^{1/2}$ *is* 1*-excessive.*

For the proof of Proposition 5.4 we need the following lemma.

Lemma 5.5. *Let* $v \in C^2(H) \cap L^1(H, \nu)$ *be such that* $v, |Dv|_{H_0^1}, \sup_{i\in\mathbb{N}} |D^2v(e_i, e_i)|$ *are bounded on* H *balls and*

$$\int_D \Big[|v(x)|\, |x|_H^2 + |Dv(x)|_{H_0^1} + |Dv(x)|_{H_0^1}\, |x|_H + \sup_{i\in\mathbb{N}} |D^2v(x)(e_i, e_i)|\Big]\, \nu(dx) < +\infty. \tag{5.4}$$

Then $v \in D(N_1)$ *and for* ν*-a.e.* $x \in H$ *one has*

$$N_1 v(x) = \sum_{i=1}^{\infty} D^2v(x)(e_i, e_i) + Dv(x)(\Delta\Psi(x)). \tag{5.5}$$

Proof. Let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi' \leq 0$, $0 \leq \chi \leq 1$, $\chi = 1$ on $(-\infty, 1]$ and $\chi = 0$ on $(2, \infty)$. For $n \in \mathbb{N}$ let

$$\chi_n(x) := \chi\left(\frac{|x|_H^2}{n^2}\right), \quad x \in H, \quad v_n := \chi_n v.$$

Then for any $x \in H$ one has

$$\begin{aligned} Dv_n(x) &= \chi_n(x)Dv(x) + v(x)D\chi_n(x) \\ &= 1_{\{|x|_H \leq 2n\}}(x)\Big[\chi_n(x)Dv(x) + \frac{2}{n^2}\, v(x)\chi'\Big(\frac{|x|_H^2}{n^2}\Big)\langle x, \cdot\rangle_H\Big]. \end{aligned} \tag{5.6}$$

Likewise for $i \in \mathbb{N}$, $x \in H$, one has

$$\begin{aligned} &D^2v_n(x)(e_i, e_i) \\ &= \chi_n(x)D^2v(x)(e_i, e_i) + v(x)D^2\chi_n(x)(e_i, e_i) + 2Dv(x)(e_i)D\chi_n(x)(e_i) \\ &= 1_{\{|x|_H \leq 2n\}}(x)\Big[\chi_n(x)D^2v(x)(e_i, e_i) \\ &+ v(x)\Big(\chi'\Big(\frac{|x|_H^2}{n^2}\Big)\frac{2}{n^2} + \frac{4}{n^4}\,\chi''\Big(\frac{|x|_H^2}{n^2}\Big)\langle x, e_i\rangle_H^2\Big) \\ &+ \frac{4}{n^2}\, Dv(x)(e_i)\chi'\Big(\frac{|x|_H^2}{n^2}\Big)\langle x, e_i\rangle_H\Big]. \end{aligned} \tag{5.7}$$

Hence $v_n \in C_b^1(H)$. Since $|\Delta\Psi|_H \in L^2(H, \nu)$ by (2.2) and

$$\int_H |x|_H^{2r}\nu(dx) \leq c_1 \int_H |x|_{L^{2r}}^{2r}\nu(dx) \leq c_2 \int_H |\Delta\Psi(x)|_H^2\nu(dx) < +\infty, \tag{5.8}$$

(cf. Remark 2.2), we see from (5.6), (5.7) that $v_n \to v$ and $N_0 v_n$ converge to the right hand side of (5.5) in $L^1(H, \nu)$ as $n \to \infty$. $\square$

Proof of Proposition 5.4. Consider the Moreau approximation Φ_ε, $\varepsilon > 0$, of Φ, i.e.,

$$\Phi_\varepsilon(x) := \min\left\{\frac{1}{2\varepsilon}\,\|y-x\|^2 + \Phi(y)\,|\,y \in H\right\}, \quad x \in H.$$

Then $\Phi_\varepsilon \in C^1(H)$, is convex and $D\Phi_\varepsilon$ is just the Yosida approximation F_ε of $(\Delta\Psi, H_\Psi)$ used in §4. Furthermore, $\Phi_\varepsilon \uparrow \Phi$ as $\varepsilon \downarrow 0$ (cf. e.g. [19, Proposition IV.1.8]).

Fix $\varepsilon, \beta > 0$ and define

$$\Phi_{\varepsilon,\beta}(x) := \int_H \Phi_\varepsilon(e^{\beta B}x + y) N_{\frac{1}{2}\,B^{-1}(e^{2\beta B}-I)}, \quad x \in H, \tag{5.9}$$

where B is as in (4.1). Then $\Phi_{\varepsilon,\beta} \in C^\infty(H)$, is convex and

$$D_H\Phi_{\varepsilon,\beta}(x) := \Delta(D\Phi_{\varepsilon,\beta}(x)) = F_{\varepsilon,\beta}(x), \quad x \in H, \tag{5.10}$$

with $F_{\varepsilon,\beta}$ as defined in (4.1). So, by the properties of $F_{\varepsilon,\beta}$ stated in §4 it follows that $D^2\Phi_{\varepsilon,\beta}$ is bounded and (4.3) implies that

$$|\Phi_{\varepsilon,\beta}(x)| \le 2C_\varepsilon(1 + |x|_H^2), \quad x \in H. \tag{5.11}$$

By (5.8), (5.11) and (4.3) it follows that all assumptions in Lemma 5.5 for $v := \Phi_{\alpha,\beta}$ are fulfilled (note that condition (5.4) indeed holds by (5.8) since $r \ge 2$). Hence $\Phi_{\alpha,\beta} \in D(N_1)$ and if we denote the right hand side of (5.5) for $v := \Phi_{\alpha,\beta}$ by $N_0\Phi_{\alpha,\beta}$ it follows that for all $x \in H$ one has

$$(1 - N_0)\Phi_{\varepsilon,\beta}(x) \le \Phi_{\varepsilon,\beta}(x) - \langle D_H\Phi_{\varepsilon,\beta}(x), \Delta\Psi(x)\rangle_H. \tag{5.12}$$

Here we used that $D^2\Phi_{\varepsilon,\beta}(x)(e_i, e_i) \ge 0$, $i \in \mathbb{N}$, since $\Phi_{\varepsilon,\beta}$ is convex. Since by (4.3) one has

$$\begin{aligned}|\langle D_H\Phi_{\varepsilon,\beta}(x), \Delta\Psi(x)\rangle_H| &\le C_\varepsilon(1 + |x|_H)|\Delta\Psi(x)|_H \\ &\le C_\varepsilon(1 + |x|_H)|\Psi(x)|_{H_0^1}\end{aligned}$$

and the right hand side is in $L^1(H, \nu)$ by (5.8) and (2.2), the right hand side of (5.12) converges to $\Phi_\varepsilon - \langle D_H\Phi_{\varepsilon,\beta}(\cdot), \Delta\Psi(\cdot)\rangle_H$ in $L^1(H, \nu)$ as $\beta \to 0$. Applying $G_1^{(1)}$ to (5.12) and letting $\beta \to 0$ we then obtain for ν-a.e. $x \in H$

$$\Phi_\varepsilon(x) \le G_1^{(1)}\Big(\Phi_\varepsilon(x) - \langle D_H\Phi_\varepsilon(x), \Delta\Psi(x)\rangle_H\Big). \tag{5.13}$$

But by (3.6) for every $x \in H_\Psi$ one has

$$\begin{aligned}|\langle D_H\Phi_\varepsilon(x), \Delta\Psi(x)\rangle_H| &= |\langle F_\varepsilon(x), F(x)\rangle_H| \\ &\le |F(x)|_H^2 = |\Psi(x)|_{H_0^1}^2.\end{aligned}$$

Since $\nu(H_\Psi) = 1$ and since $\Phi_\varepsilon + |\Psi|_{H_0^1} \in L^1(H, \nu)$, by (5.13) this implies that

$$\Phi_\varepsilon \le G_1^{(1)}\left(\Phi_\varepsilon + |\Psi|_{H_0^1}^2\right) = g \quad \nu\text{-a.e.}$$

Since $\Phi_\varepsilon \uparrow \Phi$ and $\Phi \in L^1(H,\nu)$ and since by (5.3) one has

$$\Phi(x) \geq \frac{\kappa_0}{r(r+1)}\, |x|^{r+1}_{L^{r+1}(D)}, \quad x \in H,$$

and $r+1 \geq 2$, assertion (i) follows. To prove (ii) fix $\lambda > 0$. We note that by the resolvent equation $\lambda G^{(1)}_{\lambda+1} g \leq g$, since $g \geq 0$. Hence

$$\lambda G^{(1)}_{\lambda+1} g^{1/2} \leq \frac{\lambda}{\lambda+1}\left((\lambda+1)G^{(1)}_{\lambda+1}g\right)^{1/2} = \frac{\lambda^{1/2}}{(\lambda+1)^{1/2}}\left(\lambda G^{(1)}_{\lambda+1}g\right)^{1/2} \leq g^{1/2}.$$

So, by (5.2) assertion (ii) follows. □

The last result of this section is that in some cases the Markov processes in Theorem 5.1 can even be chosen to be strong Feller on supp ν if $d=1$. More precisely, consider the following condition

$$(C1) \qquad\qquad d=1 \ \ \textit{and}\ C = (-\Delta)^{-\gamma} \ \ \textit{with}\ \gamma \in (1/2, 1].$$

Theorem 5.6. *Assume that (H1)–(H3) and (C1) hold. Then the conservative strong Markov process $\mathbb{M}$ in Theorem 5.1 can be chosen to be strong Feller on supp ν. More precisely, its semigroup satisfies $p_t f \in C_b(\text{supp}\ \nu)$ for all $f \in B_b(H)$, $t \geq 0$, and $\lim_{t\to 0} p_t f(x) = f(x)$ for all $x \in$ supp ν and all bounded Lipschitz continuous functions $f\colon H \to \mathbb{R}$. Furthermore, supp ν is an invariant set for $\mathbb{M}$ and Theorem* 5.1-(ii) *holds with $\overline{H} =$ supp ν.*

Proof. The line of argument is exactly analogous to [9]. We only mention here that the crucial estimate (4.7) in [9] can be derived in the same way in our situation here. Hypotheses 1.1(i) and 1.2(i) of [9] are not used for this. □

Remark 5.7. *(i)* We stress that according to Theorem 6.1 below we have that *supp* $\nu = H$ since (C1) implies condition (H4) below.

(ii) For the interested reader who would like to check the details from [9] for the proof of Theorem 5.6 we would like to point out an annoying misprint in [9, Lemma 5.6]. The last two lines of its statement should be replaced by " and for $t, \lambda > 0$, $x \mapsto \int_0^t \overline{p}_s f(x) e^{-\lambda s} ds$ is continuous on H_0 ".

6 Support of invariant measure

In this section, we show that any measure which is the weak limit of a sequence of invariant probability measures ν_n corresponding to the finite dimensional approximations has full support in the negative Sobolev space $H := H^{-1}(D)$ with its natural Hilbert norm $|\cdot|_H$. To this end, we obtain a uniform lower bound of ν_n-measures of any given ball in H.

Let C be a positive symmetric operator on $L^2(D)$. We assume that in addition to (H1) the operator C satisfies the following condition:

(H4) λ_k, $k \in \mathbb{N}$, *in (H1) are strictly positive and there is a Hilbert space E such that the embedding $L^2(D) \to E$ is Hilbert–Schmidt and $\sqrt{C}$ extends to an operator in $L(E, L^2(D))$ that will be denoted by the same symbol.*

A typical example is $C = (-\Delta)^{-\sigma}$, $\sigma > d/2$, and $E = H^{-\gamma}(D)$, $\gamma > d/2$.

Let W be a cylindrical Wiener process in $L^2(D)$. Then W is a continuous Wiener process with values in E. Given a function Ψ as above, we consider the mapping $F\colon\ x \mapsto \Delta(\Psi \circ x)$ on $L^2(D)$ with values in $H^{-2}(D)$.

As above, let $\{e_i\}$ be the eigenbasis of the Laplacian, let P_n be the orthogonal projection in $L^2(D)$ (and also in $H^{-1}(D)$) to the linear span E_n of $e_1, \dots, e_n$, and let $F_n := P_n F$ and $C_n := P_n\sqrt{C}$. We observe that

$$\int_D \Psi \circ x(u) \Delta x(u)\, du = -\int_D \Psi' \circ x(u) |\nabla x(u)|^2\, du \le -\kappa \int_D |x(u)|^{r-1} ||\nabla x(u)|^2\, du$$

for all $x \in E_n$. Therefore, on every subspace E_n we have $(F_n(x), x)_{L^2(D)} \to -\infty$ as $\|x\|_{L^2(D)} \to \infty$. Since F_n is continuous and dissipative on E_n, there is a diffusion process ξ_n on E_n governed (in the strong sense) by the stochastic differential equation

$$d\xi_n = F_n(\xi_n)dt + C_n dW.$$

This process has a unique invariant probability ν_n.

Theorem 6.1. *Suppose that (H1)–(H4) hold and that* $1 \le d \le 2(r+1)/(r-1)$. *Then any measure* ν *that is the limit of a weakly convergent subsequence of* $\{\nu_n\}$ *has full support in* H*, i.e., does not vanish on nonempty open sets.*

Remark 6.2. If $\nu := \mu$ where μ is the solution of (1.4) constructed in [6], then Theorem 6.1 applies to ν.

Proof of Theorem 6.1. Let us fix $x_0, x_1 \in \bigcup_{n=1}^\infty E_n$, $\varepsilon > 0$, and consider the deterministic equation

$$\begin{aligned} &y_n' = F_n(y_n) + C_n u_n^\varepsilon, \quad t \in [0,1], \\ &y_n(0) = x_0, \end{aligned} \tag{6.1}$$

where $u_n^\varepsilon \in L^2(0,1;E)$ is specified below. We consider $n \ge n_0$, where n_0 is such that $x_0, x_1 \in E_{n_0}$. By Lemma A.1 there is $u_n^\varepsilon \in L^2(0,1;E)$ such that as $n \to \infty$ one has $u_n^\varepsilon \to u^\varepsilon$ strongly in $L^2(0,1;E)$ and

$$|y_n(1) - x_1|_H \le \varepsilon. \tag{6.2}$$

Set $D_t := D \times (0,t)$. Letting $v_n^\varepsilon(t) := \int_0^t u_n^\varepsilon(s)\, ds$ we obtain

$$\xi_n(t, x_0) - y_n(t) - \int_0^t [F_n(\xi_n(s,x_0)) - F_n(y_n(s))]\, ds = C_n W(t) - C_n v_n^\varepsilon(t).$$

Set $z_n(t) := \int_0^t [F_n(\xi_n(s,x_0)) - F_n(y_n(s))]\, ds$. Then we arrive at the following representation:

$$\xi_n(t) - y_n(t) + z_n(t) = C_n W(t) - C_n v_n^\varepsilon(t).$$

Taking the inner product in H with $F_n(\xi_n(t)) - F_n(y_n(t))$ and integrating in t over $[0,1]$, we obtain

$$
\begin{aligned}
-\int_0^t \langle \xi_n(s) - y_n(s), F_n(\xi_n(s)) - F_n(y_n(s))\rangle_H \, ds + \frac{1}{2}|z_n(t)|_H^2 \\
= \int_0^t \langle C_n W(s) - C_n v_n^\varepsilon(s), F_n(\xi_n(s)) - F_n(y_n(s))\rangle_H \, ds \\
\le |C_n W - C_n v_n^\varepsilon|_{C([0,t];H)} |\Psi(\xi_n) - \Psi(y_n)|_{L^1([0,t];H)} \\
\le K_1 |W - v_n^\varepsilon|_{C([0,t];E)} |\Psi(\xi_n) - \Psi(y_n)|_{L^1([0,t];H)},
\end{aligned}
$$

where condition (H4) was employed and K_1 is a constant. Generic constants will be denoted by K with subindices. Taking into account that

$$
\langle \xi_n - y_n, F_n(\xi_n) - F_n(y_n)\rangle_H = \int_D (\xi_n - y_n)(\Psi(\xi_n) - \Psi(y_n)) \, du,
$$

we obtain for $t = 1$

$$
\begin{aligned}
-\int_{D_1} (\xi_n - y_n)(\Psi(\xi_n) - \Psi(y_n)) \, du \, ds + \frac{1}{2}|z_n(t)|_H^2 \\
\le K_1 |W - v_n^\varepsilon|_{C([0,1];E)} |\Psi(\xi_n) - \Psi(y_n)|_{L^1(0,1;H)}.
\end{aligned} \tag{6.3}
$$

On the other hand, by the Sobolev embedding theorem $L^{2d/(d+2)} \subset H$ and therefore

$$
\begin{aligned}
|\Psi(\xi_n) - \Psi(y_n)|_H \le K_2 |\Psi(\xi_n) - \Psi(y_n)|_{L^{2d/(d+2)}(D)} \\
\le K_2 \Big(\int_D \Big[|\xi_n|^{2dr/(2+d)} + |y_n|^{2dr/(2+d)} \Big] \, du \Big)^{(d+2)/(2d)}.
\end{aligned} \tag{6.4}
$$

Similarly to (6.3) we have

$$
\begin{aligned}
\int_{D_1} \xi_n \Psi(\xi_n) \, du \, ds \le \int_{D_1} x_0 \Psi(\xi_n) \, du \, ds + K_1 |W|_{C([0,1];E)} |\Psi(\xi_n)|_{L^1(0,1;H)} \\
\le K_3 \int_{D_1} |x_0| |\xi_n|^r \, du \, ds + K_4 |W|_{C([0,1];E)} \Big(\int_{D_1} |\xi_n|^{2dr/(d+2)} \, du \, ds \Big)^{(d+2)/(2d)}.
\end{aligned}
$$

Since under our assumption $2dr/(d+2) \le r+1$ we obtain

$$
\int_{D_1} |\xi_n|^{r+1} \, du \, ds \le K_5 \Big(|x_0|_{L^r(D)}^r + |W|_{C([0,1];E)}^r \Big).
$$

Similarly, we have by (6.1)

$$
\int_{D_1} |y_n|^{r+1} \, du \, ds \le K_6 \Big(|x_0|_{L^r(D)}^r + |v_n^\varepsilon|_{C([0,1];E)}^r \Big).
$$

According to (6.4) this yields

$$
\int_0^1 |\Psi(\xi_n) - \Psi(y_n)|_H \, ds \le K_7 \Big(|x_0|_{L^r(D)}^r + |W|_{C([0,1];E)}^r + |v_n^\varepsilon|_{C([0,1];E)}^r \Big).
$$

Therefore, taking into account (6.3) we obtain

$$|z_n(1)|_H^2 \le K_8 |W - v_n^\varepsilon|_{C([0,1];E)} \Big(|x_0|_{L^r(D)}^r + |W|_{C([0,1];E)}^r + |v_n^\varepsilon|_{C([0,1];E)}^r \Big),$$

which along with (6.2) gives

$$\begin{aligned} |\xi_n(1,x_0) - x_1|_{_H} &\le \varepsilon + |C_n W(1) - C_n v_n^\varepsilon(1)|_{_H} + |z_n(1)|_{_H} \\ &\le \varepsilon + K_9 |W - v_n^\varepsilon|_{C([0,1];E)}^{1/2} \Big(|x_0|_{L^r(D)}^{r/2} + |W - v_n^\varepsilon|_{C([0,1];E)}^{r/2} + 1 \Big). \end{aligned}$$

Therefore, for all $\alpha > 0$ one has

$$P\Big(|\xi_n(1,x_0) - x_1|_{_H} \ge \alpha \Big) \le P\Big(|W - v_n^\varepsilon|_{C([0,1];E)}^{1/2} \big[|x_0|_{L^r(D)}^{r/2} + |W - v_n^\varepsilon|_{C([0,1];E)}^{r/2} + 1 \big] \ge \gamma \Big),$$

where $\gamma = (\alpha - \varepsilon)/K_9$. Now let $\alpha = 2\varepsilon$ and let $B(x_1, \alpha)$ denote the closed ball of radius α in H centered at x_1. Then $B_n(x_1,\alpha) = B(x_1,\alpha) \cap E_n$ is the ball of the same radius in E_n centered at x_1 (we recall that we deal with n such that $x_1 \in E_n$). Set

$$G_n(x_0) := P\Big(|W - v_n^\varepsilon|_{C([0,1];E)}^{1/2} \big[|x_0|_{L^r(D)}^{r/2} + |W - v_n^\varepsilon|_{C([0,1];E)}^{r/2} + 1 \big] \ge \varepsilon / K_9 \Big).$$

By the invariance of the measure ν_n and the previous estimate one has

$$\nu_n\big(B_n(x_1,\alpha)\big) = \int_{E_n} P\Big(|\xi_n(1,x_0) - x_1|_H \le \alpha \Big)\, \nu_n(dx_0) \ge \int_{E_n} [1 - G_n(x_0)]\, \nu_n(dx_0).$$

Letting

$$G(x_0) := P\Big(|W - v^\varepsilon|_{C([0,1];E)}^{1/2} \big[|x_0|_{L^r(D)}^{r/2} + |W - v^\varepsilon|_{C([0,1];E)}^{r/2} + 1 \big] \ge \varepsilon / K_9 \Big),$$

we have $G(x_0) = \lim\limits_{n\to\infty} G_n(x_0)$. We recall that the measures ν_n converge weakly to ν also on the space $L^2(D)$. By convergence of u_n^ε in $L^2(0,1;E)$ we have $v_n^\varepsilon(t) \to \int_0^t u^\varepsilon(s)\, ds =: v^\varepsilon$ in $C([0,1];E)$. Therefore, the functions G_n converge to G uniformly on bounded sets in $L^2(D)$. Hence

$$\int [1 - G(x_0)]\, \nu(dx_0) = \lim_{n\to\infty} \int [1 - G_n(x_0)]\, \nu_n(dx_0).$$

This yields the estimate

$$\nu\big(B(x_1,\alpha)\big) \ge \limsup_{n\to\infty} \nu_n\big(B_n(x_1,\alpha)\big) \ge \int [1 - G(x_0)]\, \nu(dx_0).$$

It remains to observe that $G(x_0) < 1$ for every x_0. This follows by the fact that W is a nondegenerate Gaussian vector in $C([0,1];E)$, hence for any $\eta > 0$, one has $P\Big(\sup_{t\in[0,1]} |W(t) - v^\varepsilon(t)|_E < \eta \Big) > 0$. $\square$

Appendix. Approximate controllability

Let H be a separable Hilbert space, let F be an m-accretive operator on H, and let $B\colon E \to H$ be a bounded linear operator on a Hilbert space E such that $Ker(B^*) = 0$. Let $\{e_i\}$ be an orthonormal basis in H and $P_n x = \sum_{i=1}^{n}(x, e_i)e_i$ the projection to $E_n := \operatorname{span}(e_1, \dots, e_n)$. Set $F_n := P_n F|_{E_n}$.

Given $u \in L^2(0,1;E)$, let us consider the following nonlinear equation:

$$\begin{aligned} &y' = Fy + Bu, \quad t \in [0,T], \\ &y(0) = y_0. \end{aligned} \tag{6.5}$$

We also consider finite dimensional equations

$$\begin{aligned} &y_n' = F_n y_n + P_n Bu, \quad t \in [0,T], \\ &y_n(0) = P_n y_0. \end{aligned} \tag{6.6}$$

It was proved in [4] that equation (6.5) is approximately controllable, i.e., given $\varepsilon > 0$ and $y_0, y_1 \in \overline{D(F)}$, there is $u \in L^2(0,1;E)$ such that $|y(1) - y_1|_{_H} \le \varepsilon$. Here we prove a sharper result in terms of the approximating problem (6.6).

Lemma 6.3. *Given $\varepsilon > 0$ and $y_0, y_1 \in \overline{D(F)}$, there exists $u_n^\varepsilon \in L^2(0,T;E)$ such that $|y_n(T) - P_n y_1|_{_H} \le \delta(\varepsilon)$, $\lim_{n\to\infty} u_n^\varepsilon = u^\varepsilon$ in $L^2(0,T;E)$ and $|y^{u_\varepsilon}(T) - y_1|_{_H} \le \delta(\varepsilon)$, where y^u is the solution to (6.5) and $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$.*

Proof. It suffices to prove our claim for $y_0, y_i \in D(F)$. We fix n and $\varrho > 0$ and consider the differential inclusion

$$z_n' \in F_n z_n - \varrho \operatorname{sgn}(z_n - P_n y_1) \quad \text{a.e. } t \in [0,T] \tag{6.7}$$

$$z_n(0) = P_n y_0. \tag{6.8}$$

It is known (see [2]) that (6.7) has a unique solution $z_n \in W^{1,\infty}([0,T], E_n)$ and

$$z_n'(t) = F_n z_n(t) - \varrho \operatorname{sgn}(z_n(t) - P_n y_1) \quad \text{a.e. on } [0,1], \tag{6.9}$$

where $\operatorname{sgn} w$ is the unit vector $w/|w|$ if $w \neq 0$, sgn0 is the unit ball $\{h \in H_n\colon\ |h_n|_{_H} < 1\}$. Therefore,

$$\frac{d}{dt}|z_n(t) - P_n y_1|_{_H} + \varrho \le |P_n F y_1|_{_H} \quad \text{a.e. } t > 0. \tag{6.10}$$

Hence, whenever $\varrho > |Fy_1|_{_H} + |y_0 - P_n y_1| T^{-1}$, we have $|z_n(t) - P_n y_1| = 0$ for all $t \ge T$. We set $v_n(t) \in -\varrho \operatorname{sgn}(z_n(t) - P_n y_1)$, where $z_n'(t) = F_n z_n(t) + v_n(t)$ a.e. on $[0,T]$. By (6.10) we see that $t \mapsto |z_n(t) - P_n y_1|_{_H}$ is decreasing on $[0,T]$, hence there exists $T_0 \in (0,T]$ such that $|z_n(t) - P_n y_1|_{_H} > 0$ for all $t \in [0,T_0)$ and therefore

$$v_n(t) = -\varrho \frac{z_n(t) - P_n y_1}{|z_n(t) - P_n y_1|_{_H}} \quad \text{for } t \in [0,T_0]. \tag{6.11}$$

On the other hand, by (6.9) we have

$$z_n'(t) = \Big(F_n z_n(t) - \varrho \operatorname{sgn}\big(z_n(t) - P_n y_1\big)\Big)^0,$$

where $(D)^0$ stands for the minimal section of a set D. We have therefore

$$v_n(t) = \mathrm{Proj}_{B(0,\varrho)} F_n(P_n y_1) \quad \text{for } t \in [T_0, T]. \tag{6.12}$$

By (6.11) and (6.12) we conclude that as $n \to \infty$ we have convergence $v_n \to v$ in $L^2(0,T;H)$. It is clear that $z_n \to z$ in $C([0,T];H)$, where $z'(t) = Fz + v$ a.e. on $[0,T]$, $z(0) = y_0$, $z(T) = y_1$. Next, letting $B_n := P_n B$, we define u_n^ε to be the point where the function $|B_n u - v_n|^2_{L^2(0,T;H)} + \varepsilon |u|^2_{L^2(0,T;E)}$ attains its minimum. We have

$$B_n^*(B_n u_n^\varepsilon - v_n) + \varepsilon u_n^\varepsilon = 0. \tag{6.13}$$

Finally, we define u^ε to be the point where the function $|Bu - v|^2_{L^2(0,T;H)} + \varepsilon |u|^2_{L^2(0,T;E)}$ attains its minimum. We have

$$B^*(Bu^\varepsilon - v) + \varepsilon u^\varepsilon = 0. \tag{6.14}$$

It follows by (6.13) and (6.14) that $u_n^\varepsilon \to u^\varepsilon$ in $L^2(0,T;E)$ as $n \to \infty$. Moreover, since $|Bu^\varepsilon - v|^2_{L^2(0,T;H)} + \varepsilon |u^\varepsilon|^2_{L^2(0,T;E)} \le |v|^2_{L^2(0,T;H)}$ we have by (6.14) that $Bu^\varepsilon - v \to 0$ weakly in $L^2(0,T;H)$ as $\varepsilon \to 0$. Replacing $\{u^\varepsilon\}$ by a suitable sequence of the arithmetic means of u^{ε_i} we may assume that $Bu^\varepsilon \to v$ in the norm of $L^2(0,T;H)$. Then we see that

$$|B_n u_n^\varepsilon - v_n|_{L^2(0,T;H)} \le \eta_1(1/n) + \eta_2(\varepsilon) + C_1 |u^\varepsilon - u_n^\varepsilon|_{L^2(0,T;H)} + |B_n u^\varepsilon - Bu^\varepsilon|_{L^2(0,T;H)},$$

where $\eta_i(s) \to 0$ as $s \to 0$, $i = 1,2$. Then we obtain $|B_n u_n^\varepsilon - v_n|_{L^2(0,T;H)} \le 4\eta_2(\varepsilon) =: \delta(\varepsilon)$ for all $n \ge N(\varepsilon)$. □

We remark that this proof remain valid if F is quasi-m-dissipative, i.e., $F + \gamma I$ is m-dissipative for some $\gamma > 0$. In addition, F may be multivalued.

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